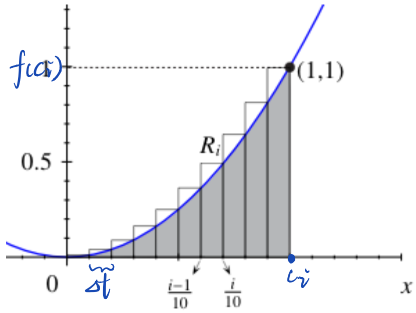
积分

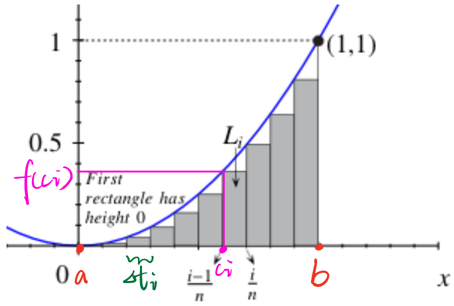
一 原理：Riemann Sum.

$S = \sum_{i=1}^{n} f(c_i)\Delta t_i \qquad \Delta t_i = \frac{b-a}{n}$



R-H R.S

$c_i = a + i\frac{b-a}{n}$

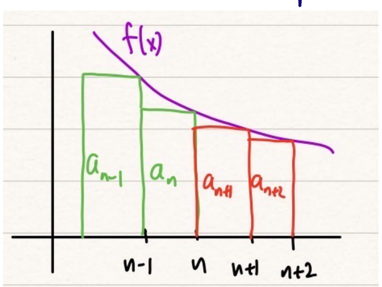


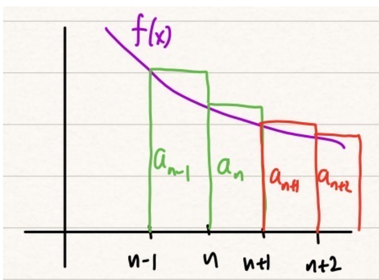
L-H R.S

$c_i = a + (i-1)\cdot\frac{b-a}{n}$

integrability theorem: $\int_a^b f(t)\,dt = \lim_{n\to\infty} S_n = \lim_{n\to\infty}\sum_{i=1}^{n} f(c_i)\Delta t_i$

error estimate for conv. series.





求 # terms 使 error < err

1. 证 $\sum a_n$ conv.

2. $S - S_k = \int_k^{\infty} f \le err$

error: $R_n = L - S_n = \sum_{i=n+1}^{\infty} a_i$ （所有红色面积之和）

$$\int_{n+1}^{\infty} f(x)\,dx \le R_n \le \int_n^{\infty} f(x)\,dx$$

in AST. $|R_n| = |S - S_n| < a_{n+1}$

$$S_n + \int_{n+1}^{\infty} f(x)\,dx \le L \le S_n + \int_n^{\infty} f(x)\,dx$$

$\int_n^{\infty}$ 真实值 整个线下面积 $\int_n^{\infty}$ 真实值

n: # terms

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AVT: $f_{av} = \frac{1}{b-a}\int_a^b f(t)\,dt \qquad \exists\, a \le c \le b. \quad f(c) = f_{av}$

FTC:

I. $\frac{d}{dx}\int_{a(x)}^{b(x)} f(t)\,dt = f(b(x))\cdot b'(x) - f(a(x))\cdot a'(x)$

II. $\int_a^b f(t)\,dt = F(b) - F(a)$

一 计算

1. 换底

$$\int_b^a f(g(x))\, g'(x)\, dx \qquad u = g(x) \quad \text{(取较复杂的)}$$

$$= \int_{g(b)}^{g(a)} f(u) \frac{du}{dx}\, dx$$

⊕ 要记得改变.

2. 分部

$$\int u \frac{dv}{dx}\, dx = uv - \int v \frac{du}{dx}\, dx.$$

简单的. xe^x. $\sqrt{\ln x}$

3. Inverse Trignometric Substitution.

$$1 - \sin^2 x = \cos^2 x$$
$$\tan^2 x + 1 = \sec^2 x$$
$$\sec^2 x - 1 = \tan^2 x$$

4. Partial

$$\frac{1}{(x-a)^n} = \frac{A_1}{x-a} + \cdots + \frac{A_n}{(x-a)^n} \qquad (n \text{ terms})$$

记得长除.

5. $\int^{\infty} / \int_{\infty}$ Improper integral

检查 verticle asymptote

Yes — No

$x = c$ 写成 $\lim_{n\to\infty}$ 的形式

$$\int_a^b = \int_a^{c^-} + \int_{c^+}^b$$

(写成 $\lim_{x\to c^{\pm}}$ 形式)

可用 换元/分部.

$\ln x^k = k \ln x$

$y = \ln(f(x)) \qquad \frac{dy}{dx} = \frac{f'(x)}{f(x)}$

$\log_n ab = \log_n a + \log_n b.$

$\int f'(x) e^{f(x)}\, dx = e^{f(x)} + c$

$\int a^{kx+b}\, dx = \frac{1}{k} \cdot \frac{a^x}{\ln a} + c$

$\int t e^{-t}\, dt \qquad u = t$

$p^{\ln n} = n^{\ln p}.$

6. differential equation.

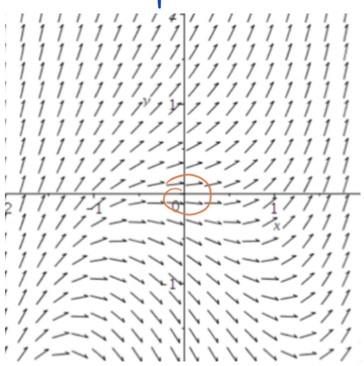
- def. linear : $a_n(x)y^{(n)} + \cdots + a_0(x)y = f(x)$ ← order n阶导

  separable : $y' = f(x)g(y)$

  $y' + y^2 = \cos x$ 是 quadratic. 不是 linear.

  $\sin y$. e^y 不是 first order.

- direction field.



找 $y'=0$ 的点. → 线是 —

$y'=1$ 的点 → 线是 /

- first-order linear diff eq. FOLDE

① 确认 linear. 写作 $y' = f(x)y - g(y)$.

② $I(x) = e^{\int f(x)\,dx}$

③ $y = \dfrac{\int g(x) I(x)\,dx}{I(x)}$

- separable diff eq

① 找解. ($\frac{dy}{dx} = 0$)

② y一边. x一边 $f(y)\frac{dy}{dx} = g(x)$ $f(y)dy = g(x)dx$

③ 两边积分

④ 尽量找 $y(x)$

一应用

1. 求面积

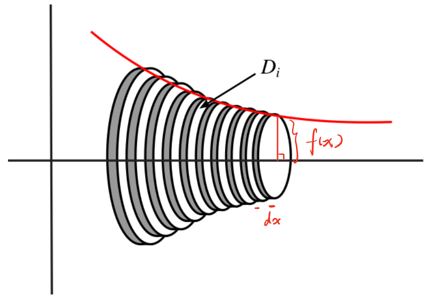
求交点 → 分块积分（上-下）

2. 求旋转体积

① 竖切 $V=\pi r^2 h$.

→ 单线 Disk $r=f(x)$ $h=dx$

→ 双线 Washers $f^2(x)-g^2(x)$.

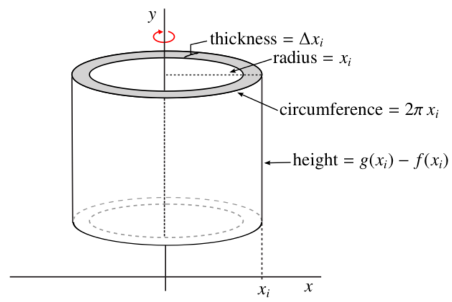


② 横切 $dV = C\cdot h\cdot \text{thickness} = 2\pi r\cdot h\cdot \text{thickness}$

→ 双线

$r=y$ $h=f(y)-g(y)$ thickness : dy

x $f(x)-g(x)$ dx



Volume $= 2\pi\, x_i\,(g(x_i) - f(x_i))\,\Delta x_i$

3. arclength (S)

$$S=\int_a^b \sqrt{1+(f'(x))^2}\,dx$$

4. exponential growth & decay

$$P(t)=P(0)\,e^{kt}$$

5. logic growth

$\frac{dT}{dt}=k(T-T_s)$ T_s : surrounding

$$T(t)=(T_0-T_s)e^{kt}+T_s$$

6. Newton's law of cooling

$\frac{dP}{dt}=kP(M-P)$ M : max P

$P>0$ $P(t)=M\frac{Ce^{Mkt}}{1+Ce^{Mkt}}$

$P<0$ $P(t)=M\frac{Ce^{Mkt}}{Ce^{Mkt}-1}$

$\int^{\infty}$ improper

1. comparison test

$$0 \le f \le g$$

f: div. g: conv

✿ 注意上下界取值.

$$\int_1^{\infty} \frac{2+e^{-x}}{x}\,dx$$

当 $x>0$ 时, $e^{-x}<1$.

✿ 有时无法直接用 comparison.

先用 improper integral

先用 comparison

$$\int_1^{\infty} \frac{\sin(x)}{x}\,dx$$

2. p-test

$p>1$ $\quad \int_1^{\infty} \frac{1}{x^p} = \frac{1}{p-1}$

$p<1$ $\quad \int_0^{1} \frac{1}{x^p} = \frac{1}{1-p}$

3. ACT

$\int_0^{\infty} |f|\,dx$ conv $\Rightarrow \int_a^{\infty} f\,dx$ conv.

Series

都可

✿ $p>0$, $x\to\infty$

$\ln(x)^p \ll x^p \ll p^x \ll x^x$

1. comparison test

$$0 \le a_n \le b_n$$

a_n: div; b_n: conv

2. LCT

$$L = \lim_{n\to\infty} \frac{a_n}{b_n}$$

✿ 需满足 a_n, b_n 同号

✿ 选 b_n 使 a_n, b_n 中 n 最高系数相同.

$L=0$: a_n div, b_n conv

$0<L<\infty \to$ con $\Leftrightarrow$ con

$L=\infty$: a_n conv, b_n div

(ab) 3. ratio

$$L = \lim_{n\to\infty} \left|\frac{a_{n+1}}{a_n}\right|$$

✿ $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^n = e$

$L>1$ $\quad a_n$ div

$L=1$ $\quad$ unknown

$L<1$ $\quad a_n$ conv absolutely

(ab) 4. root

$$L = \lim_{n\to\infty} |a_n|^{\frac{1}{n}}$$

$L>1$ $\quad a_n$ div

$L=1$ $\quad$ unknown

$L<1$ $\quad a_n$ conv absolutely

5. p-series test

$p>1 \Leftrightarrow \sum_{n=1}^{\infty} \frac{1}{n^p}$ conv

证 conv

1. geometric $\quad \sum_{n=0}^{\infty} Ar^n = \frac{A}{1-r}$

2. arithmetic thm II. $\quad \sum_{n=j}^{\infty} a_n$ conv $\Leftrightarrow \sum_{n=1}^{\infty} a_n$ conv

3. integral test

   $f(x)=a_n$. pos. cont. dec.

   $\int_1^{\infty} f(x)\,dx$ conv. $\Leftrightarrow \sum_1^{\infty} a_n$ conv

4. AST $\quad$ 适用于带 $(-1)^n$ in a_n

   a_n: $\lim = 0$. decr. pos $\Rightarrow \sum_{n=1}^{\infty} (-1)^{n-1} a_n$ conv.

(ab) 5. ACT $\quad \sum |a_n|$ conv $\Rightarrow \sum a_n$ conv ab'

证 div

1. divergence test $\lim_{n\to\infty} a_n \neq 0$
2. harmonic series. $\sum_{n=1}^{\infty} \frac{1}{n}$

判断 conv/div：反例优先考虑之前 $n\geq 0$. $\sum \frac{(-1)^n}{\sqrt{n}}$

- div
- conv
  - cond conv $\rightarrow \sum |a_n|$ div $\sum a_n$ conv. $\rightarrow$ eg. $\sum_{2}^{\infty} \frac{(-1)^n}{n\ln n}$, $\sum_{n=2}^{\infty} (-1)^n \cdot \frac{1}{n}$
  - abs conv $\rightarrow \sum |a_n|$ conv $\sum a_n$ conv.

☆ 记得 check end point

power series $\sum_{0}^{\infty} a_n (x-a)^n$

$|x-a|<R$

R: ↓ I:

$L = \lim_{n\to\infty} \left| \frac{a_{n+1}}{a_n} \right|$

$0<L<\infty$	$R=\frac{1}{L}$	$\longrightarrow$	$R\geq |x-a|$
$L=0$	$R=\infty$	$\longrightarrow$	$x\in\mathbb{R}$.
$L=\infty$	$R=0$.	$\longrightarrow$	$x=a$.

运算

1. $(f\pm g)(x) = \sum_{n=0}^{\infty} (a_n+b_n)(x-a)^n$ $R\geq \min\{R_f, R_g\}$ $I=I_f\cap I_g$
   $(x-a)^m f(x) = \sum_{n=0}^{\infty} a_n (x-a)^{m+n}$ $R=R_f$ $I=I_f$.

2. 求导&积分 $f(x)=\sum_{0}^{\infty} b_n(x-a)^n$ $R>0$
   $f'(x) = \sum_{1}^{\infty} n b_n (x-a)^{n-1}$
   $\int f(x) = \sum_{0}^{\infty} \frac{b_n (x-a)^{n+1}}{n+1} + c$.

   $\rightarrow$ R不变. I变.

* $\sum_{n=0}^{\infty} x^n = \frac{1}{1-x}$

* find series representation for ...
  1. 找 $f(x)$ 与 $\frac{1}{1-x}$ 的关系 (求导/积分/ substitution)
  2. 带入.

特殊形式 (a=0 in 一些情况) $\rightarrow$ Taylor Poly

$$e^x = \sum_{k=0}^{n} \frac{x^k}{k!}$$

$$\sin x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k+1}}{(2k+1)!}$$

$$\cos x = \sum_{k=0}^{n} \frac{(-1)^k x^{2k}}{(2k)!}$$